The magnetic field produced at a focus of a current-carrying conductor in the shape of a conic section

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We determine the magnetic field at the focus of a conic section due to a current along the conic section. For a given current I, it is shown that this magnetic field is the same and equal to $\mu_0 I/2p$ for all conics with the same semilatus rectum p.
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I. INTRODUCTION

The calculation of magnetic fields in textbooks using the Biot–Savart law is limited mainly to current-currying conductors consisting of linear or circular sections. In rare cases the field at the focus of a current-carrying elliptical or parabolic conductor is calculated.[1] Recently the field of a more general plane current loop was considered.[2] The purpose of this paper is to demonstrate an interesting common property of conic sections regarding the magnetic field produced at one of the foci of a current-carrying conic section.

II. ANALYSIS

We consider a current-carrying conductor in the shape of a conic section (a circle, ellipse, parabola, or hyperbola). Polar coordinates (r,θ) are used to describe the curves, with a focus of the conic section taken to be the origin O, and the axis of symmetry of the curve coinciding with the polar axis (see Fig. 1). There is a constant current I in the conductor in the direction of increasing θ. A current element $Id\mathbf{r}$ with position vector $\mathbf{r}$ relative to O produces a magnetic field $d\mathbf{B}$ at O, which is given by the Biot–Savart law as

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{r}\times d\mathbf{r}}{r^3}. \tag{1}$$

For the geometry and current in Fig. 1, the direction of $d\mathbf{B}$ is normal to the plane of the figure and outward.

In polar coordinates the general equation of conic sections (position vector of a point on the curve) is[3]

$$\mathbf{r} = \frac{p}{1+e\cos\theta}\hat{\mathbf{r}}, \tag{2}$$

where $\hat{\mathbf{r}}$ is a unit vector from O to the current element, the length p is the semilatus rectum of the conic, and e is its eccentricity ($e=0$ for a circle, $0<e<1$ for an ellipse, $e=1$ for a parabola, and $e>1$ for the left branch of a hyperbola).

If we differentiate $\mathbf{r}$ with respect to θ, we obtain

$$\frac{d\mathbf{r}}{d\theta} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\hat{\mathbf{r}} + \frac{p}{1+e\cos\theta}\frac{d\hat{\mathbf{r}}}{d\theta}. \tag{3}$$

The quantity

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \mathbf{e}_\theta \tag{4}$$

is the unit vector in the direction of increasing θ. Therefore

$$\hat{\mathbf{r}}\times\frac{d\mathbf{r}}{d\theta} = \frac{p}{1+e\cos\theta}\hat{\mathbf{r}}\times\mathbf{e}_\theta \tag{5}$$

or

$$\hat{\mathbf{r}}\times\frac{d\mathbf{r}}{d\theta} = \frac{p}{1+e\cos\theta}\hat{\mathbf{z}}, \tag{6}$$

where $\hat{\mathbf{z}}=\hat{\mathbf{r}}\times\mathbf{e}_\theta$ is the unit vector normal to both $\hat{\mathbf{r}}$ and $\mathbf{e}_\theta$, and is out of the plane of the curve in Fig. 1.

Equations (2) and (6) and the definition $\mathbf{r}=r\hat{\mathbf{r}}$ give

$$\mathbf{r}\times\frac{d\mathbf{r}}{d\theta} = \left(\frac{p}{1+e\cos\theta}\right)^2\hat{\mathbf{z}} \tag{7}$$

so that the ratio in Eq. (1) becomes

$$\frac{\mathbf{r}\times d\mathbf{r}}{r^3} = \left(\frac{1}{p}+\frac{e}{p}\cos\theta\right)d\theta\hat{\mathbf{z}}. \tag{8}$$

The total magnetic field at O for a circle, ellipse, or parabola is found by integrating Eq. (1) from $\theta=-\pi$ to π

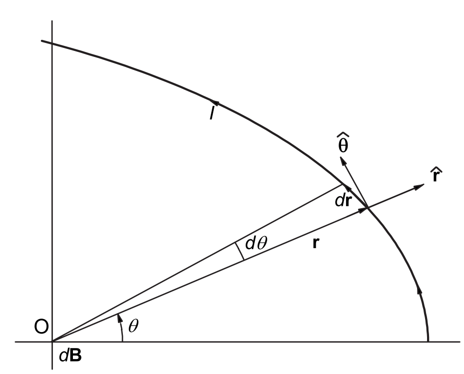

Fig. 1. A current element $Id\mathbf{r}$ on a conic section and the magnetic field $d\mathbf{B}$ it produces at O, the focus of the curve.

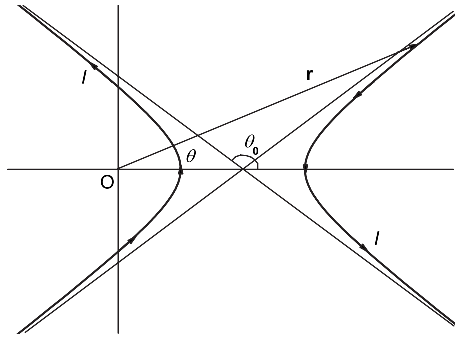

Fig. 2. The two branches of a hyperbola.

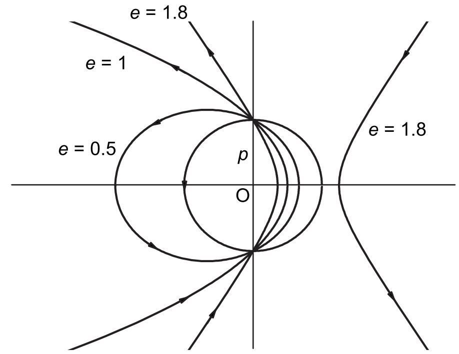

Fig. 3. Conic sections with a common focus O and the same value of the latus rectum $2p$.

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\pi}^{\pi}\left(\frac{1}{p} + \frac{e}{p}\cos\,\theta\right)d\theta. \tag{9}$$

If instead of the equation of a conic such as that in Eq. (2), we had used the general form

$$\mathbf{r} = r(\theta)\hat{\mathbf{r}} \tag{10}$$

for a curve in plane polar coordinates, Eq. (9) would have the more general form

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\oint \frac{d\theta}{r}, \tag{11}$$

derived in Ref. 2. We perform the integration in Eq. (9) and obtain

$$\mathbf{B} = \frac{\mu_0 I}{2p}\hat{\mathbf{z}} \tag{12}$$

for a circle, ellipse, or parabola.

The hyperbola has to be treated separately because it has two branches between its two asymptotes, which form angles of θ_0 and $\pi - \theta_0$ with its axis of symmetry (see Fig. 2). The value of θ_0 is given by $\cos\,\theta_0 = -1/e$ because this value of $\cos\,\theta$ makes r infinite. The field produced by the left branch of the hyperbola is

$$\mathbf{B}_+ = 2\frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_0^{\theta_0}\left(\frac{1}{p} + \frac{e}{p}\cos\,\theta\right)d\theta = \frac{\mu_0 I}{2\pi p}(\theta_0 + e\,\sin\,\theta_0)\hat{\mathbf{z}}. \tag{13}$$

Equation (2) produces negative values of r for $\theta_0 \le \theta < 2\pi - \theta_0$. We can keep r positive if we change the sign of the right-hand side of Eq. (2) and can limit θ to the range $-(\pi - \theta_0) \le \theta \le \pi - \theta_0$ for the right branch of the hyperbola if we write $\cos(\theta - \pi)$ instead of $\cos\,\theta$ in Eq. (2). The right branch of the hyperbola is then given by

$$\mathbf{r} = \frac{p}{e\,\cos\,\theta - 1}\hat{\mathbf{r}} \tag{14}$$

for $-(\pi - \theta_0) \le \theta \le \pi - \theta_0$.

For the current direction shown in Fig. 2, the magnetic field produced at O by the right branch of the hyperbola is

$$\mathbf{B}_- = -2\frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_0^{\pi-\theta_0}\left(-\frac{1}{p} + \frac{e}{p}\cos\,\theta\right)d\theta$$

$$= \frac{\mu_0 I}{2\pi p}(\pi - \theta_0 - e\,\sin\,\theta_0)\hat{\mathbf{z}}. \tag{15}$$

The total magnetic field produced by a hyperbolic conductor at its focus is therefore

$$\mathbf{B} = \mathbf{B}_+ + \mathbf{B}_- = \frac{\mu_0 I}{2p}\hat{\mathbf{z}}, \tag{16}$$

the same as for the other conic sections. The field at the right-hand focus of the hyperbola is also given by Eq. (16).

We conclude that conic sections with the same semilatus rectum p, carrying the same current I, produce the same magnetic field $\mu_0 I/2p$ at their foci. A group of such conic sections is shown in Fig. 3 with their common focus at O.

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